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Optimal Markdown and Credit Decisions in a Two-Warehouse Integrated Inventory Model

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Abstract

In the case of perishable products whose demand is time-sensitive, the decisions on order and sale policies have to be appropriately made to minimize the costs of product decay. The markdown policy, widely adopted by retailers, is a well-known sales policy that effectively manages the revenue and cost of perishable inventories. A significant decision in this policy is to determine an optimal time to mark down the price and boost demand. Moreover, trade credits serve as short-term financing through which the seller sets a deadline for the buyer to pay for the purchased products. Once given a trade credit, the buyer is enabled to increase the order and make a better profit out of sales before meeting the credit deadline. If the optimal quantity of the order exceeds the capacity of the retailer's warehouse, another warehouse should be rented. In this context, the present study aims to simultaneously examine markdown and credit decisions for perishable products in a two-warehouse inventory model. For this purpose, a set of theorems is used to obtain optimal sales and purchase decisions under integrated decision-making, and the proposed model is solved with numerical examples. Finally, sensitivity analyses are conducted to evaluate the effects of parameter variations on decision variables. The numerical results demonstrate the significance of a combination of credit financing and markdown policies for perishable inventories. It has also been shown that time-diminishing demand is beneficial for supply chains. The proposed model can help managers make optimal decisions on trade credit and markdown policies.

Keywords: Two-warehouse, Trade credit, Integrated inventory, Discount, Markdown.

1 | Introduction

Ordering and sale policies are effective in optimal income and cost management. These factors are of significance, especially in the case of perishable inventories. Products such as fruits, vegetables, and medicines degenerate over time [1–3]. They can be kept or used only until their expiration date; they cease to be usable as that date is reached [4]. In the United States, for example, the food articles wasted due to perishing comprise

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about 10% of the whole food supplied at the retail level [5]. As customers generally prefer to buy fresh or new products, the demand for this kind of product decreases over time as the perishable products age. A short shelf life and a time-sensitive demand make ordering and selling perishable goods challenging [6]. Using a successful marketing plan to manage demand might assist retailers in properly ordering and managing sales. The markdown policy is a well-known marketing strategy that reduces prices for less fresh items. Of course, it is possible to purposefully reduce the price of outdated items to draw in buyers. The mark down policy aims to entice customers who would not purchase because they would rather pay a lower price.

Moreover, firms adopt different policies to pay for goods and services. They include a) the policy of prepayment, by which the customer must pay the price before receiving the goods; b) the policy of cash payment, by which the price is paid right after the product is received; and c) the policy of delayed payment (i.e., credit shopping), by which the seller gives the buyer a short-term interest-free loan [7]. Due to its advantages for both the seller and the buyer under fund limitations, the policy of giving trade credits has been widely favored in financial transactions over the last few decades. A trade credit policy allows buyers to get the intended goods without paying on the spot and to gain interest by investing the cash in a share market or a bank [8]. This has made the credit policy very popular in financial transactions. For instance, Walmart Company asks its suppliers for trade credits [9]. A credit is a subsidy that contributes to the buyer's flexibility in budget allocation and the reduction of inventory holding costs [10].

Consequently, trade credits allow buyers to place a considerably increased order such that the optimal quantity of the order exceeds the Owned Warehouse (OW) capacity [11]. In this case, the buyer has to rent another warehouse for the surplus goods [12]. Generally, Rented Warehouses (RW) are suitable places owing to their good conditions, low rate of perishing, and good storage facilities, but they may impose a high storage cost. Therefore, the inventory in this warehouse is drawn on first.

Given the above discussion, managing a deteriorating inventory appropriately and dealing with the buyer's budget deficit are two basic issues in many realistic supply chains. For example, in a food supply chain, small retailers ask big producers to let them settle their bills after a given period. A trade credit provides flexibility for the retailers to collect revenue during the permissible delay and make partial or full payments. This persuades them to increase their orders and keep products in owned and RW. In addition, these retailers use a markdown policy as a sales strategy to cope with the cost of deteriorating inventory. The markdown policy attracts more customers and promotes sales volume by offering a discount on the price.

As *Table 1* demonstrates, many studies have considered trade credit as a financing scheme in inventory and supply chain models. In addition, many researchers have paid attention to markdown policies' impact on retailers' profit. However, the integration of these issues in a two-warehouse system has not been investigated yet. Therefore, the identified research gap and the aforementioned real-life example motivate this study.

In this regard, the present study is the first attempt to explore the effects of markdown and trade credit policies on a two-warehouse integrated inventory of perishable goods. The study seeks to practically present supply chain policies by answering the following questions:

I. How long should the optimal trade credit and cycle be to maximize supply chain profits?
II. When should the retailer discount the price to boost demand and get the maximum profit?
III. What quantity of goods should be stored in a rented warehouse?

The rest of this paper is organized into a few sections. Section 2 is dedicated to a review of the literature in the field. Sections 3 and 4 present the study's assumptions and the corresponding mathematical equations. A set of theorems and their solution algorithms are given in Section 5. Then, Section 6 evaluates the proposed framework through numerical examples. Section 7 presents sensitivity analyses and discusses managerial insights. Finally, Section 8 closes up the paper with the conclusion of the study and some recommendations for future research.

2|Literature Review

This study addresses two lines of research in the field—the first concerns perishable goods' income and cost management under a markdown policy. Markdown pricing is one of the most common policies of sale enhancement. It is practiced to supply goods to customers before they expire and become useless. Pashigian [13] was one of the first researchers who worked on the markdown policy for fashion goods in a two-period model. Xu et al. [14] showed that the optimal policies of dynamic pricing and markdown for perishable products could affect the choice of a cooperation contract (i.e., a wholesale price contract or a consignment revenue-sharing contract) by the seller and the buyer. Jeihouni et al. [15] formulated an inventory system with a non-zero ending for non-instantaneous deteriorating items to make optimal pricing and markdown decisions. They believed that the price would drop before the quality of the items deteriorated and that demand was a function of both price and stock.

The markdown strategy was proposed by Srivastava and Gupta [16] to increase demand for non-instantaneously degrading commodities, where demand was determined by price and time. Assuming that demand is sensitive to both price and time, Kaya and Polat [6] calculated the optimal price reduction points and the ideal time for markdown pricing in a perishable inventory model. In a perishable inventory scenario, Zabihi and Bafruei [17] looked at the markdown strategy, where demand was an exponential function of time and a linear function of price. Şen [18] presented optimal policies of markdown pricing for two rival sellers whose products were perishable and substitutable.

Hu et al. [19] introduced the optimal policies of markdown pricing in an inventory system with customer behavior considered. Chua et al. [20] formulated four different models by considering the effects of markdown policy and customer behavior, as well as uncertain demand for perishable goods. Some researchers have compared fixed and markdown pricing strategies for perishable goods with the assumption that demand is affected by the quality and price of the goods. For example, to compare fixed and markdown pricing strategies, Wang et al. [21] considered two different scenarios: a) the markdown time is already determined, but the optimal price has to be calculated, and b) the price is a parameter, but the markdown time is a variable. Chen et al. [22] used the game theory approach to compare the seller's and buyer's optimal decisions under fixed and markdown pricing strategies. It is to be noted that an optimal pricing strategy is selected for markdown costs.

Chen and Chen [23] examined markdown policies in three scenarios for common customers, quality-oriented customers, and price-oriented customers. They did it assuming that demand was a function of the price and quality of products. Mousavi et al. [24] proposed a Stackelberg model to determine the optimal hotel pricing in the tourism industry, with discount incentives considered. Liang and Lin [25] and Abbasi Siar et al. [26] analyzed the impact of discount promotions on customers' purchase behavior. Wu and Xu [27] investigated different collaboration strategies in a sustainable supply chain where the unsold products were discounted at the end of the selling period.

Qiu et al. [28] presented two-period pricing divided into regular and discount periods for customers with heterogeneous behaviors in the presence of a quick response possibility for inventory stock-out. Wang et al. [29] optimized pricing and inventory decisions under three sales policies, while the products remaining from the first period were sold at a lower price in the second period. Wang et al. [30] compared the effects of discount promotion and coupon promotion on the seller's profit in a two-period model, considering the promotion was offered only in the first period.

The second line of research in the literature regards trade credits. Goyal [31] was the first to formulate an EOQ model under a trade credit. Shah [32] examined an uncertain inventory model for perishable goods at times when delayed payment is permissible. Then, Jamal et al. [33] added some backlogging assumptions to an EOQ model concerning credit deadlines. Several researchers have investigated crediting policies in integrated inventory models. One may refer to references [34–40] in this case. Setak et al. [34] formulated a

supply chain coordination model under trade credit and markdown policies to determine optimal ordering and selling decisions. Jaggi et al. [10] studied a two-warehouse inventory model that was credited for trading on non-instantaneous deteriorating goods.

Similarly, Chakrabarty et al. [35] considered the inflation, time value of the money, and trade credit period in a two-warehouse inventory system for perishable items, assuming that a shortage was allowed. Gupta et al. [36] also presented an inventory system with two warehouses for perishable products, assuming that the deterioration rate was a function of time and that a shortage was allowed. In a similar study, Khan et al. [37] developed a two-warehouse inventory system where the products perish faster in the main warehouse than in the rented one. The retailer had to pay off its debts partly in equal installments and partly in Cash On Delivery (COD).

Khan et al. [38] and Das et al. [39] devised a two-warehouse inventory model assuming that the trade credit period depended on the order quantity and that a shortage was allowed. Mashud et al. [40] suggested the two-warehouse inventory model, considering multiple partial advance payments and partial trade credits with advertisement-dependent demand for perishable items.

Murmu et al. [41] compared the impacts of partial and full trade credits on a deteriorating inventory under an inspection policy in a two-store system. Peng et al. [42] suggested an effective algorithm for optimizing replenishment and delivery decisions in a multi-warehouse environment under a trade credit. Under a conditional trade credit, Yang [50] examined optimal perishable inventory policies, considering a limited warehouse capacity.

Panda et al. [43] presented credit financing in a two-store deteriorating inventory model with advertisement, price, and stock-sensitive demand, as well as allowable shortages. Momena et al. [44] extended all unit quantity discount inventory with time-dependent holding cost and trade credit in a two-warehouse system. Yang [45] examined the effect of inflation on a deteriorating inventory model under a two-warehouse setup and a partial backlogging demand. Table 1 highlights the novelty of this research as compared to the other studies conducted in the field.

Table 1. A comparison of this research and some similar studies.

Author(S)	Trade Credit	Markdown	Two-Warehouse	Deterioration	Demand Type	Model Type
Xu et al. [14]	-	✓	-	✓	Price and time	Decentralized
Jeihouni, et al.[15]	-	✓	-	✓	Price and stock	EOQ
Wang et al. [21]	-	✓	-	✓	Quality and price	EOQ
Chen et al.[22]	-	✓	-	✓	Quality and price	Decentralized
Chen and Chen [23]	-	✓	-	✓	Quality and price	EOQ
Dai and Wang [46]	✓		-	✓	Stochastic	Integrated
Setak et al. [34]	✓	✓	-	✓	Time and price	Coordination
Gupta et al. [36]	✓	-	✓	✓	Constant	EOQ
Khan et al. [37]	-	-	✓	✓	Constant	EOQ
Das et al. [39]	✓	-	-	-	Constant	Integrated
Mashud et al. [40]	✓	-	✓	✓	Advertisement and price	EOQ
Murmu et al. [41]	✓	-	✓	✓	Advertisement and price	EOQ
Yang [47]	✓	-	✓	✓	Expiration date	EOQ
Panda et al. [43]	✓	-	✓	-	Advertisement, price and stock	EOQ
Momena et al. [44]	✓	-	✓	✓	Advertisement and price	EOQ
Yang [45]	✓	-	✓	✓	Constant	EOQ
This paper	✓	✓	✓	✓	Time and price	Integrated

As seen in *Table 1*, none of the reviewed studies have simultaneously investigated the effects of trade credit and markdown policies in a two-warehouse inventory system. This gap in the literature can be filled by evaluating the effects of these two parameters on the seller and buyer's optimal decisions in a two-warehouse

integrated inventory model. The present study is novel in a few aspects. Firstly, it formulates the seller and buyer's objective functions in six cases concerning the possible relationships among the markdown time, time of the inventory depletion in the RW, and trade credit period. Secondly, certain theorems and efficient algorithms are used to determine the optimal values of the buyer's decisions (i.e., sale cycle length, markdown time, and inventory depletion time in the RW) and the seller's decisions (i.e., credit period and wholesale price). Thirdly, the parameters are made to vary to evaluate the optimality of each case.

3 | Statement of the Problem

This study is based on a two-echelon integrated supply chain in which a food producer sells a kind of perishable food to a supermarket. The demand is assumed to be time- and price-sensitive. The food producer supplies Q_0 units of the food product to the supermarket at the price of v per unit. After receiving Q_0 units, the supermarket stores Y units in its warehouse and the remaining S units in the rented one. The products begin to perish at the rate of θ in both warehouses. Since the holding cost is higher in the RW than in the OW, the inventory of the RW is used first. During $[0 \;\; t_e]$, the inventory of the RW decreases due to the demand and perishability, but that of the OW changes only due to the perishing of the goods. At time t_e, the inventory in the RW ends up, and then the OW begins to be withdrawn. Within $[t_e \;\; T]$, the inventory in the OW is affected by the demand and the perishing of the goods.

By giving a trade credit to the supermarket, the food producer allows it to pay for the goods after the duration M. Until the trade credit deadline, the supermarket deposits the sale revenues in a bank and receives interest at the rate of R_e. Once the deadline is reached, the supermarket has to pay the purchase cost of the products left in the warehouse, along with a charge at the rate of R_i.

The demand rate declines over time since the product is perishable and demand is time-sensitive. This has negative effects on the revenue. Therefore, an incentive mechanism is used to recoup new customers and increase sales volume. The supermarket resorts to the common markdown policy to entice the customers to buy. Each selling period $[0 \;\; T]$ is comprised of a non-markdown period and a markdown period. In the non-markdown period $[0 \;\; t_s]$, the product is sold at the price of p_1 and the demand of $N(p_1, t)$. During the markdown period $[t_s \;\; T]$However, the price is reduced to p_2 at time t_s and the demand is changed to $N(p_2, t)$. Considering the markdown time (t_s) and the inventory depletion time in the RW (t_e), the two scenarios are as follows:

Scenario 1: $t_s \leq t_e$

This scenario expresses that the supermarket discounts the price before the RW inventory is depleted (*Fig. 1a*).

Scenario 2: $t_s > t_e$

In this scenario, the supermarket discounts the price of the products after the inventory depletion in the RW. It means the markdown policy is not applied to the inventory stored in the RW (*Fig. 1b*).

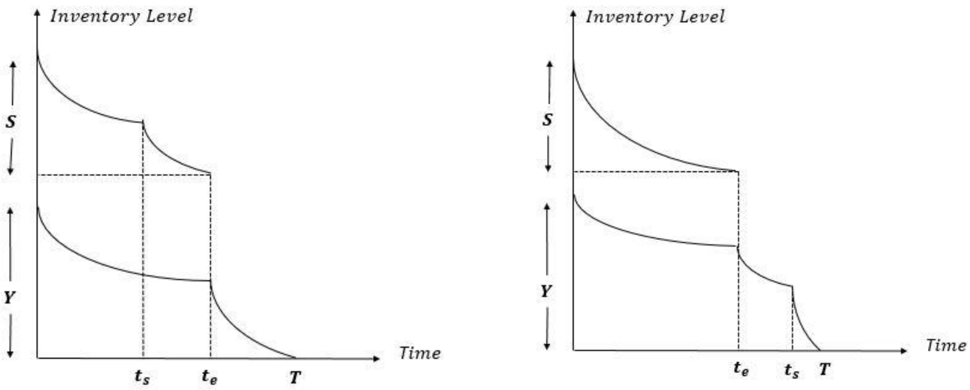


Fig. 1. The system behavior in two scenarios: 1. $t_s \leq t_e$ 2. $t_e < t_s$.

3.1 | Assumptions and Symbols in the Mathematical Model

3.1.1 | Assumptions

1. The delivery of the product is instantaneous.
2. The demand is time- and price-sensitive:

$$N(p_i, t) = B - \alpha p_i - \lambda t. \quad (1)$$

3. The products perish at the rate of θ.
4. No shortage is allowed.
5. The wholesale price (v) is a function of the trade credit period:

$$v = g + l.M, \quad (2)$$

$g > 0$ represents the procurement cost in the absence of a credit period, and $l > 0$.

6. Due to being given credit, the producer incurs investment opportunity costs.

3.1.2 | Symbols

The symbols used in this study are presented in *Table 2*.

Table 2. Notations.

Symbol	Description	Symbol	Description
R_e	Rate of interest received per year	A	Supermarket's ordering cost per order
R_i	Rate of interest charged per year	O	Food producer's ordering cost per order
R_b	Investment opportunity cost rate per year	h	Holding cost per unit in OW
B	Initial demand	H	Holding cost per unit in RW
λ	Coefficient for the sensitivity of demand to time	c	Food producer's production cost per unit
p_2	Marked down price	α	Coefficient for the sensitivity of demand to price
p_1	Prime price	T	Sales cycle length (decision variable) in month
t_e	Inventory depletion time in the RW (decision variable)	t_s	Markdown time (decision variable) in month
S	Maximum inventory level of the RW	Y	Capacity of the OW
v	Wholesale price (decision variable) per unit	M	Trade credit period (decision variable) in month
$N(p_1, t)$	Demand rate before markdown	Q_0	Supermarket's order quantity
$N(p_2, t)$	Demand rate after markdown	Q_1	Amount of the products sold before markdown
Ts	Food producer's total profit	Q_2	Amount of the products sold after markdown
Tr_j	Supermarket's total profit in case j	Tsr_j	Supply chain total profit in case j

3.2 | Mathematical Model Formulation

In this section, the formulations of the mathematical model are derived under two scenarios. For this purpose, the inventory level in two warehouses at time t is calculated according to each scenario:

Scenario 1: $t_s \leq t_e$

It is noted that the inventory level is changed based on the demand and a fixed deterioration rate. In this scenario, the price is discounted before the RW inventory depletion. Then, the RW inventory level before the markdown time $Ir_1(t)$ and after markdown time $Ir_2(t)$ at time t can be defined with the differential *Eq. (3) and Eq. (4)*

$$\frac{dIr_1(t)}{dt} = -N(p_1, t) - \theta Ir_1(t), \quad 0 \leq t < t_s. \tag{3}$$

$$\frac{dIr_2(t)}{dt} = -N(p_2, t) - \theta Ir_2(t), \quad t_s \leq t \leq t_e. \tag{4}$$

Considering the two boundary conditions of $Ir_1(0) = S$ and $Ir_2(t_e) = 0$, the above differential equations yield.

$$Ir_1(t) = \int_0^t -N(p_1, u)e^{\theta(u-t)}du + Se^{-\theta t}. \tag{5}$$

$$Ir_2(t) = \int_t^{t_e} -N(p_2, u)e^{\theta(u-t)}du. \tag{6}$$

The value of S can be calculated with the continuity condition $Ir_1(t_s) = Ir_2(t_s)$.

During $[0\ \ t_e]$, the OW inventory level is affected only by deterioration. Therefore, the inventory level $Iy_1(t)$ in the OW at time t is obtained through *Eq.(7)* as follows:

$$\frac{dIy_1(t)}{dt} = -\theta Iy_1(t), \quad 0 \leq t \leq t_e. \tag{7}$$

The boundary condition $Iy_1(0) = Y$ yields the following:

$$Iy_1(t) = Ye^{-\theta t}. \tag{8}$$

In $[0\ \ t_e]$, the demand and deterioration make the OW inventory decrease. The level of this inventory is denoted by $Iy_2(t)$ and determined by *Eq. (9)*

$$\frac{dIy_2(t)}{dt} = -N(p_2, t) - \theta Iy_2(t), \quad t_e \leq t \leq T. \tag{9}$$

With $Iy_2(T) = 0$, there is:

$$Iy_2(t) = \int_t^T N(p_2, u)e^{\theta(u-t)}du \ \ t_e \leq t \leq T. \tag{10}$$

Scenario 2: $t_s > t_e$

In the second scenario, the price of the products is reduced after the RW inventory is depleted. Then, that inventory is consumed with demand $N(p_1, t)$. The inventory level $Ir_1{}'(t)$ in the RW at time t is formulated as *Eq. (11)*

$$\frac{dIr_1{}'(t)}{dt} = -N(p_1, t) - \theta Ir_1{}'(t), \quad 0 \leq t \leq t_e. \tag{11}$$

With $Ir_1{}'(t_e) = 0$, the following is yielded by solving the above equation:

$$Ir_1{}'(t) = \int_t^{t_e} N(p_1, u)e^{\theta(u-t)}du. \tag{12}$$

Using $Ir_1{}'(0) = S'$ leads to

$$S' = \int_0^{t_e} N(p_1, u)e^{\theta u}du. \tag{13}$$

The inventory level in the OW is obtained through *Eq. (14)* to *Eq. (16)*:

$$\frac{dIy_1{}'(t)}{dt} = -\theta Iy_1{}'(t), \qquad 0 \le t \le t_e. \tag{14}$$

$$\frac{dIy_2{}'(t)}{dt} = -N(p_1,t) - \theta Iy_2{}'(t), \quad t_e \le t \le t_s \tag{15}$$

$$\frac{dIy_3{}'(t)}{dt} = -N(p_2,t) - \theta Iy_3{}'(t), \quad t_s \le t \le T. \tag{16}$$

With boundary condition $Iy_1{}'(0) = Y'$ and continuity condition $Iy_2{}'(t_s) = Iy_3{}'(t_s)$, the following relations are yielded:

$$Iy_1{}'(t) = Y'e^{-\theta t}. \tag{17}$$

$$Iy_2{}'(t) = \int_t^{t_s} N(p_1,u)e^{\theta(u-t)}du + Iy_3{}'(t)e^{-\theta(t_s-t)}. \tag{18}$$

$$Iy_3{}'(t) = \int_t^{T} N(p_2,u)e^{\theta(u-t)}du. \tag{19}$$

According to inventory level relationships, the supermarket component costs in each cycle are expressed as follows:

The holding cost for keeping the inventory in both warehouses is equal to:

$$HO = \begin{cases} HO_1 = h\left(\int_0^{t_s} Ir_1(t)dt + \int_{t_s}^{t_e} Ir_2(t)dt\right) + H\left(\int_0^{t_e} Iy_1(t)dt + \int_{t_e}^{T} Iy_2(t)dt\right), & t_s \le t_e, \\ HO_2 = h\left(\int_0^{t_e} Ir_1{}'(t)dt\right) + H\left(\int_{t_e}^{t_s} Iy_2{}'(t)dt + \int_{t_s}^{T} Iy_3{}'(t)dt\right), & t_s > t_e. \end{cases} \tag{20}$$

The quantity of the order placed to the food producer is as follows:

$$Q_0 = \begin{cases} Q_0{}^1 = Y + S, & t_s \le t_e, \\ Q_0{}^2 = Y' + S', & t_s > t_e. \end{cases} \tag{21}$$

The cost of ordering from the supermarket is A, and the purchase cost equals vQ_0.

During trade credit, the supermarket deposits the selling revenue in a bank account with the interest rate of R_e to receive more profit. After the trade credit expiration, the supermarket is charged with the interest rate of R_i for the goods unsold.

The possible relationships among markdown time (t_s), RW inventory depletion time (t_e) and trade credit (M) can be various cases, each of which involves a specific received or charged interest:

$$\begin{cases} \text{Case 1: } M \le t_s \le t_e, \\ \text{Case 2: } t_s \le M \le t_e, \\ \text{case 3: } t_s \le t_e \le M, \\ \text{case 4: } M \le t_e \le t_s, \\ \text{case 5: } t_e \le M \le t_s, \\ \text{case 6: } t_e \le t_s \le M. \end{cases}$$

In the following, the received and charged interests of each case are calculated

Case 1: $M \le t_s \le t_e$

In case 1, the price is reduced after the trade credit deadline and before the inventory depletion in the RW. The received interest IE_1 and the charged interest IP_1 are as follows:

$$IE_1 = R_e\, p_1 \int_0^M \int_0^t N(p_1,t)dudt. \tag{22}$$

$$\mathrm{IP_1 = R_i\, v\left(\int_M^{t_s} Ir_1(t)dt + \int_{t_s}^{t_e} Ir_2(t)dt + \int_M^{t_e} Iy_1(t)dt + \int_{t_e}^{T} Iy_2(t)dt\right).} \tag{23}$$

Case 2: $\mathrm{t_s \le M \le t_e}$

In case 2, the trade credit expires before the end of the inventory in the RW and after the price reduction. The received interest $\mathrm{IE_2}$ and the charged interest $\mathrm{IP_2}$ are equal to:

$$\mathrm{IE_2 = R_e(p_1(\int_0^{t_s}\int_0^{t} N(p_1,t)dudt + \int_{t_s}^{M}\int_0^{t_s} N(p_1,t)dudt) + p_2(\int_{t_s}^{M}\int_{t_s}^{t} N(p_2,t)dudt))} \tag{24}$$

$$\mathrm{IP_2 = R_i\left(v\left(\int_M^{t_e} Ir_2(t)dt + \int_M^{t_e} Iy_1(t)dt + \int_{t_e}^{T} Iy_2(t)dt\right)\right).} \tag{25}$$

Case 3: $\mathrm{t_s \le t_e \le M}$

In case 3, the depletion of the RW inventory occurs before the trade credit expiration and after the price reduction. The received and charged interests are equal to

$$\mathrm{IE_3 = R_e(p_1(\int_0^{t_s}\int_0^{t} N(p_1,t)dudt + \int_{t_s}^{M}\int_0^{t_s} N(p_1,t)dudt) + p_2(\int_{t_s}^{M}\int_{t_s}^{t} N(p_2,t)dudt)).} \tag{26}$$

$$\mathrm{IP_3 = R_i v\left(\int_M^{T} Iy_2(t)dt\right).} \tag{27}$$

Case 4: $\mathrm{M \le t_e \le t_s}$

In this case, markdown pricing is only done for the products stored in the OW, and the trade credit expires before the end of the RW inventory.

$$\mathrm{IE_4 = R_e p_1\left(\int_0^{M}\int_0^{t} N(p_1,t)dudt\right).} \tag{28}$$

$$\mathrm{IP_4 = R_i v\left(\int_M^{t_e} Ir_1'(t)dt + \int_M^{t_e} Iy_1'(t)dt + \int_{t_e}^{t_s} Iy_2'(t)dt + \int_{t_s}^{T} Iy_3'(t)dt\right).} \tag{29}$$

Case 5: $\mathrm{t_e \le M \le t_s}$

In this case, it is assumed that markdown is only applied to the products stored in the OW and that the trade credit expires after the depletion of the RW inventory and before the markdown.

$$\mathrm{IE_5 = R_e p_1(\int_0^{M}\int_0^{t} N(p_1,t)dudt).} \tag{30}$$

$$\mathrm{IP_5 = R_i v\left(\int_M^{t_s} Iy_2'(t)dt + \int_{t_s}^{T} Iy_3'(t)dt\right).} \tag{31}$$

Case 6: $\mathrm{t_e \le t_s \le M}$

In case 6, the price of the products only stored in the OW is reduced, and the trade credit deadline is after the markdown.

$$\mathrm{IE_6 = R_e\left(p_1(\int_0^{t_s}\int_0^{t} N(p_1,t)dudt + \int_{t_s}^{M}\int_0^{t_s} N(p_1,t)dudt) + p_2\int_{t_s}^{M}\int_{t_s}^{t} N(p_2,t)dudt\right).} \tag{32}$$

$$\mathrm{IP_6 = R_i v\left(\int_M^{T} Iy_3'(t)dt\right).} \tag{33}$$

Therefore, the profit function of the supermarket in $[0\ \ T]$ is calculated as *Eq.(34)*:

$$\mathrm{Tr_i(t_e,t_s,T) = \frac{1}{T}(\,p_1Q_1 + p_2Q_2 - vQ_0 - A - HO + IE_i - IP_i),\quad i = 1 \dots 6.} \tag{34}$$

The first and second parts refer to sale revenue before and after markdown; the third and fourth parts equal the ordering and holding costs; and the fifth and sixth parts point to the received and charged interests.

In addition, the profit function of the food producer is as *Eq.(35)*.

$$\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}[(\mathrm{v}-\mathrm{w})\mathrm{Q}_0-\mathrm{O}-\mathrm{R}_\mathrm{b}\mathrm{M}\mathrm{v}\mathrm{Q}_0]. \tag{35}$$

The first part concerns the sale revenue, the second part refers to the cost of ordering, and the third part regards the cost of investment opportunity, which the supermarket has to incur due to being given a trade credit.

4 | Integrated Inventory Model

Every sector has a certain level of strategic coordination between buyers and sellers. An integrated inventory model is a useful tool in this situation for managing chain liquidity and maximizing revenues for all participants in the supply chain. To implement the optimal strategy for the whole chain, the supply chain suggested in this study offers its members the chance to engage in long-term strategic collaboration. In this instance, the supply chain's overall profit is as follows:

$$\mathrm{Tsr}(\mathrm{M},\mathrm{v},\mathrm{t_e},\mathrm{t_s},\mathrm{T})=\begin{cases}\mathrm{Tsr}_1=\mathrm{Tr}_1(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^1-\mathrm{A}-\mathrm{HO}_1+\mathrm{IE}_1-\mathrm{IP}_1+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^1-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^1) \\ \qquad \mathrm{M}\le\mathrm{t_s}\le\mathrm{t_e} \\ \mathrm{Tsr}_2=\mathrm{Tr}_2(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^1-\mathrm{A}-\mathrm{HO}_1+\mathrm{IE}_2-\mathrm{IP}_2+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^1-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^1) \\ \qquad \mathrm{t_e}\le\mathrm{M}\le\mathrm{t_e} \\ \mathrm{Tsr}_3=\mathrm{Tr}_3(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^1-\mathrm{A}-\mathrm{HO}_1+\mathrm{IE}_3-\mathrm{IP}_3+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^1-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^1) \\ \qquad \mathrm{t_s}\le\mathrm{t_e}\le\mathrm{M} \\ \mathrm{Tsr}_4=\mathrm{Tr}_4(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^2-\mathrm{A}-\mathrm{HO}_2+\mathrm{IE}_4-\mathrm{IP}_4+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^2-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^2) \\ \qquad \mathrm{M}\le\mathrm{t_e}\le\mathrm{t_s} \\ \mathrm{Tsr}_5=\mathrm{Tr}_5(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^2-\mathrm{A}-\mathrm{HO}_2+\mathrm{IE}_5-\mathrm{IP}_5+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^2-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^2) \\ \qquad \mathrm{t_e}\le\mathrm{M}\le\mathrm{t_s} \\ \mathrm{Tsr}_6=\mathrm{Tr}_6(\mathrm{t_e},\mathrm{t_s},\mathrm{T})+\mathrm{Ts}(\mathrm{M},\mathrm{v})=\frac{1}{\mathrm{T}}(\mathrm{p}_1\mathrm{Q}_1+\mathrm{p}_2\mathrm{Q}_2-\mathrm{v}{\mathrm{Q}_0}^2-\mathrm{A}-\mathrm{HO}_2+\mathrm{IE}_6-\mathrm{IP}_6+(\mathrm{v}-\mathrm{w}){\mathrm{Q}_0}^2-\mathrm{O}-\mathrm{R_b}\mathrm{M}\mathrm{v}{\mathrm{Q}_0}^2) \\ \qquad \mathrm{t_e}\le\mathrm{t_s}\le\mathrm{M}\end{cases} \tag{36}$$

The optimal profit of the supply chain is given as:

$$\mathrm{Tsr}^*=\mathrm{Max}(\mathrm{Max\,Tsr}_1,\mathrm{Max\,Tsr}_2,\mathrm{Max\,Tsr}_3,\mathrm{Max\,Tsr}_4,\mathrm{Max\,Tsr}_5,\mathrm{Max\,Tsr}_6). \tag{37}$$

5 | Solution Method

This section presents optimal solutions for the food producer and the supermarket. To this end, the concavity and unimodality theories are applied to find an optimal solution [48]. When we have $\frac{\partial^2\mathrm{G(t)}}{\partial\mathrm{t}^2}\le 0, \mathrm{G(t)}$ is a differentiable continuous in t, and if $\frac{\partial\mathrm{G}}{\partial\mathrm{t}}=0$, then t is an extremum point. Since the profit functions of supply chain members are subject to constraints, all possible interior and boundary solutions are considered to find a unique, globally optimal solution. Therefore, the interior and boundary points in each case are calculated separately, and the profits of the feasible points are compared. The feasible point with the maximum profit is selected as the optimal point.

5.1 | Optimal Solution under the First Case ($0\le\mathrm{M}\le\mathrm{t_s}\le\mathrm{t_e}<\mathrm{T}$)

$Tsr_1=Tr_1(t_e,t_s,T)+Ts(M,v)$,

s.t.

$M\le t_s\le t_e$.

Theorem 1.

1. For the given values of M, $\mathrm{t_s}$ and $\mathrm{t_e}$, Tsr_1 is pseudo-concave to T.
2. For the given values of M, $\mathrm{t_s}$ and T, Tsr_1 is concave to $\mathrm{t_e}$.

3. For the given values of M, t_e and T, Tsr_1 is concave to t_s.
4. For the given values of t_s, t_e and T, Tsr_1 is concave to M.

Proof: refer to *Appendix A*.

According to *Theorem 1*, Tsr_1 has an extremum point. As the equations $\frac{\partial Tsr_1}{\partial T} = 0$, $\frac{\partial Tsr_1}{\partial t_e} = 0$, $\frac{\partial Tsr_1}{\partial t_s} = 0$ and $\frac{\partial Tsr_1}{\partial M} = 0$ are simultaneously solved, the interior point $(M_1, t_{e1}, t_{s1}, T_1)$ obtained for the Tsr_1 is feasible if $0 \le M \le t_s \le t_e < T$ is satisfied.

$$\begin{aligned}\frac{\partial Tsr_1}{\partial T} = {} & (-6vR_i((\alpha p_2 + \lambda t_e - B)\theta - \lambda)e^{-\theta(M - t_e)} - 6v\alpha\theta R_i(p_1 - p_2)e^{-\theta(M - t_s)} + 6(R_i v \\ & + H)(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{\theta(T - t_e)} - 6\theta^2 Y(R_i v \\ & + H)e^{-\theta t_e} + 6e^{-\theta M} Y R_i \theta^2 v - 6((\alpha p_2 + \lambda t_e - B)\theta - \lambda)((MR_i v + c)\theta \\ & + h)e^{\theta t_e} - 6\theta(p_1 - p_2)\alpha((MR_b v + c)\theta + h)e^{\theta t_s} + (6MYR_i v + (-6{p_2}^2 t_s \\ & + 3{p_1}^2(M^2 R_e + 2t_s))\alpha + ((-3T^2 - 3{t_s}^2)p_2 + p_1(M^3 R_e + 3{t_s}^2))\lambda \\ & + 6Bp_2 t_s - 3p_1(M^2 R_e + 2t_s)B + 6Yc + 6A + 6O)\theta^3 + ((((-6p_2 t_s \\ & + 6p_1(t_s - M))R_i + 6p_1 MR_b)\alpha - 3i(M^2 + T^2)\lambda + 6BM(R_i - R_b))v \\ & + ((-6Ht_e + 6h(t_e - t_s))p_2 + 6p_1(ht_s + c))\alpha + ((-3T^2 - 3{t_e}^2)H \\ & + 3{t_e}^2 h)\lambda + (6Ht_e - 6ht_e - 6c)B + 6HY)\theta^2 + ((6i(p_1 + p_2)\alpha + ((6t_e \\ & + 6M)R_i - 6MR_b)\lambda - 12BR_i)v + (6Hp_2 + 6hp_1)\alpha + (6Ht_e - 6c)\lambda \\ & - 6B(H + h))\theta - 6(2R_i v + H + h)\lambda)/(6\theta^3 T^2).\end{aligned} \tag{38}$$

$$\frac{\partial Tsr_1}{\partial t_s} = \frac{\left(\alpha R_i v e^{-\theta(M - t_s)} + ((MR_b v + c)\theta + h)\alpha e^{\theta t_s} + ((-p_1 - p_2)\theta - R_i v - h)\alpha + \theta(-\lambda t_s + B)\right)(p_1 - p_2)}{\theta T}. \tag{39}$$

$$\begin{aligned}\frac{\partial Tsr_1}{\partial t_e} = {} & (-R_i\theta v(-\alpha p_2 - \lambda t_e + B)e^{-\theta(M - t_e)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v \\ & + H)e^{\theta(T - t_e)}) - (R_i v + H)Y\theta^2 e^{-\theta t_e} - \theta(-\alpha p_2 - \lambda t_e + B)((MR_b v \\ & + c)\theta + h)e^{\theta te} + (h - H)(-\alpha p_2 - \lambda t_e + B)\theta - \lambda(R_i v + H))/(\theta^2 T).\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{\partial Tsr_1}{\partial M} = {} & (2i((-\alpha p_2 - \lambda t_e + B)\theta + \lambda)(v\theta - g)e^{-\theta(M - t_e)} - 2\theta R_i(v\theta - g)(p_1 - \\ & p_2)\alpha e^{-\theta(M - t_s)} - 2g((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i e^{\theta(T - t_e)} + 2\theta^2 R_i(v\theta - g)Ye^{-\theta M} + \\ & 2\theta^2 YgR_i e^{-\theta t_e} - 4(Mg + d/2)\theta R_b((-\alpha p_2 - \lambda t_e + B)\theta + \lambda)e^{\theta t_e} + 4(Mg + \\ & d/2)\theta^2 R_b(p_1 - p_2)\alpha e^{\theta ts} + (-M^2\lambda p_1 R_e - 2M\alpha {p_1}^2 R_e + 2BMp_1 R_e - 4MYR_b g - \\ & 2YR_b d\theta^3) + ((((4p_1 M - 2p_1 t_s + 2p_2(t_s - T))\alpha + 3\lambda M^2 - T^2\lambda - 4BM + \\ & 2BT)g - 2d(-M\lambda - \alpha p_1 + B))R_i + 4(Mg + d/2)R_b(-\alpha p_1 + B))\theta^2 + ((((-2p_1 - \\ & 2p_2)\alpha - 2t_e\lambda - 4M\lambda + 4B)g - 2\lambda d)R_i + 4(Mg + d/2)\lambda b)\theta + 4\lambda gR_i)/(2\theta^3 T).\end{aligned} \tag{41}$$

The boundary solutions are then examined according to the following constraints.

$a^1)$ $0 < M = t_s < t_e < T$,	$a^2)$ $0 < M < t_s = t_e < T$,	$a^3)$ $0 < M = t_s = t_e < T$,
$a^4)$ $M = 0, 0 < t_s = t_e < T$,	$a^5)$ $M = 0, 0 < t_s < t_e < T$,	$a^6)$ $0 < M < t_s = t_e < T$.
$a^7)$ $0 = M = t_s, t_s < t_e < T$.		

For the case a^1, $M = t_s$ is placed in *Eq. (38)* to *Eq. (-40)*, the solution of $(t_{e11}, t_{s11}, T_{11})$ is feasible only if it is true in $0 < t_s < t_e < T$. For the case a^2, $t_s = t_e$ is placed in *Eq. (38), Eq. (39)* and *Eq. (41)*, the solution of $(M_{12}, t_{s12}, T_{12})$ is feasible only if it satisfies the constraint $0 < M < t_s < T$. For the case a^3, $M = t_s = t_e$ is placed in *Eq. (38)* and *Eq. (39)*, the solution of (t_{s13}, T_{13}) is feasible only if $0 < t_{s13} < T_{13}$. For the case a^4, $M = 0$ and $t_s = t_e$ are placed in *Eq. (38)* and *Eq. (39)*, the solution of (t_{s14}, T_{14}) is feasible only if the $0 < t_s < T$ is met. For the case a^5, *Eq. (38)* to *Eq. (40)* should be solved with regard to $M = 0$. The point $(0, t_{e15}, t_{s15}, T_{15})$ is feasible if constraint $0 < t_s < t_e < T$ is not violated. For the case a^6, *Eq. (38), Eq. (39)* and *Eq. (41)* should be solved, and the feasibility of the point $(M_{16}, t_{s16}, T_{16})$ in the relation $0 < M، 0 < t_s < T$ needs to be verified. The case a^7 postulates *Eq. (38)* and *Eq. (40)*, which should be solved concerning $M = t_s = 0$. The point $(0, t_{e17}, T_{17})$ is feasible if $0 < t_e < T$. Finally, the optimal solution in case 1 is achieved by comparing the profits of possible feasible points.

5.2 | Optimal Solution under the Second Case ($0 < t_s \le M \le t_e < T$)

$Tsr_2 = Tr_2(t_e, t_s, T) + Ts(M, v)$,

s.t.

$t_s \le M \le t_e < T$.

Theorem 2.

1. For the given values of M, t_s and t_e, Tsr_2 is pseudo-concave to T.
2. For the given values of M, t_s and T, Tsr_2 is concave to t_e.
3. For the given values of M, t_e and T, Tsr_2 is concave to t_s.
4. For the given values of t_s, t_e and T, Tsr_2 is concave to M.

The proof for *Theorem 2* is the same as that for *Theorem 1*.

According to *Theorem 2*, Tsr_2 has an extremum point. As the equations $\frac{\partial Tsr_2}{\partial T} = 0$, $\frac{\partial Tsr_2}{\partial t_s} = 0$, $\frac{\partial Tsr_2}{\partial t_e} = 0$ and $\frac{\partial Tsr_2}{\partial M} = 0$ are simultaneously solved, the interior point $(M_2, t_{e2}, t_{s2}, T_2)$ obtained for the Tsr_2 is feasible if it is true in $0 \le t_s \le M \le t_e < T$.

$$\begin{aligned}\frac{\partial Tsr_2}{\partial T} = {} & (-6((\alpha p_2 + \lambda t_e - B)\theta - \lambda)R_i v e^{-\theta(M - t_e)} + 6(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 \\ & + B)\theta + \lambda)(R_i v + H)e^{\theta(T - t_e)} - 6\theta^2 Y(R_i v + H)e^{-\theta t_e} \\ & + 6e^{-\theta M} Y R_i \theta^2 v - 6((\alpha p_2 + \lambda t_e - B)\theta - \lambda)((MR_b v + c)\theta \\ & + h)e^{\theta te} - 6(p_1 - p_2)\theta\alpha((MR_b v + c)\theta + h)e^{\theta t_s} + (((M^3 R_e - 3MR_e {t_s}^2 \\ & + 2R_e {t_s}^3 - 3T^2 - 3{t_s}^2)p2 - 2p_1 {t_s}^2(t_s R_e - 3/2MR_e - 3/2))\lambda + 6MYR_b v \\ & + ((3M^2 R_e - 6MR_e t_s + 3R_e {t_s}^2 - 6t_s){p_2}^2 - 3{p_1}^2 t_s(-2MR_e \\ & + R_e t_s - 2))\alpha - 3B(M^2 R_e - 2MR_e t_s + R_e {t_s}^2 - 2t_s)p_2 + 3p_1 s(-2MR_e \\ & + R_e t_s - 2)B + 6Yc + 6A + 6O)\theta^3 + ((-3R_i(M^2 + T^2)v \\ & + (-3T^2 - 3{t_e}^2)H + 3{t_e}^2 h)\lambda - 6((-R_b p_1 + R_i p_2)\alpha - B(R_i - R_b))Mv \\ & + ((-6ht_s + 6t_e(h - H))p_2 + 6p_1(ht_s + c))\alpha + (6Ht_e - 6ht_e - 6c)B \\ & + 6HY)\theta^2 + ((((6R_i - 6R_b)M + 6R_i t_e)v + 6Ht_e - 6c)\lambda + \\ & 12R_i(\alpha p_2 - B)v + (6Hp_2 + 6hp_1)\alpha - 6B(H + h))\theta - 6(2R_i v + H + h)\lambda)/(6\theta^3 T^2).\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{\partial Tsr_2}{\partial t_s} = {} & -(-((MR_b dv + c)\theta + h)\alpha e^{\theta t_s} + ((-p_1 - p_2)\alpha - \lambda t_s + B)(-1 + (t_s - M)R_e)\theta + \\ & \alpha h)(p_1 - p_2)/(\theta T).\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{\partial Tsr_2}{\partial t_e} = {} & (-R_i\theta v(-\alpha p_2 - \lambda te + B)e^{-\theta(M - t_e)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)(iv + H)e^{\theta(T - t_e)} - \\ & (R_i v + H)Y\theta^2 e^{-\theta t_e} - \theta(-\alpha p_2 - \lambda t_e + B)((MR_b v + c)\theta + h)e^{\theta t_e} + (h - \\ & H)(-\alpha p_2 - \lambda t_e + B)\theta - \lambda(R_i v + H))/(\theta^2 T).\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial Tsr_2}{\partial M} = {} & (2R_i((-\alpha p2 - \lambda t_e + B)\theta + \lambda)(v\theta - g)e^{-\theta(M - t_e)} - 2g((-T\lambda - \alpha p_2 + B)\theta + \\ & \lambda)R_i e^{\theta(T - t_e)} + 2\theta^2 R_i(v\theta - g)Ye^{-\theta M} + 2\theta^2 YgR_i e^{-\theta t_e} - 4(Mg + d/2)\theta R_b((-\alpha p_2 - \lambda t_e + B)\theta + \\ & \lambda)e^{\theta t_e} + 4(Mg + d/2)\theta^2 R_b(p_1 - p_2)\alpha e^{\theta t_s} + (-4MYR_b g - R_e(p_2 M^2 + {t_s}^2(p_1 - p_2))\lambda + \\ & 2p2R_e(-\alpha p_2 + B)M - 2dYR_b + 2R_e((-p_1 - p_2)\alpha + B)t_s(p_1 - p_2))\theta^3 + \\ & ((((3M^2 - T^2)\lambda - 4(-\alpha p_2 + B)(M - T/2))R_i + 4MR_b(-\alpha p_1 + B))g + 2((M\lambda + \\ & \alpha p_2 - B)R_i + R_b(-\alpha p_1 + B))d)\theta^2 + ((((-2t_e - 4M)\lambda - 4\alpha p_2 + 4B)R_i + 4\lambda MR_b)g + \\ & 2\lambda d(R_b - R_i))\theta + 4\lambda gR_i)/(2\theta^3 T).\end{aligned} \tag{45}$$

Then, the boundary solutions are examined according to the following constraints:

b^1) $0 < t_s = M < t_e < T$, b^2) $0 < t_s = M = t_e < T$, b^3) $0 < t_s < M = t_e < T$,

b^4) $0 = t_s < M = t_e < T$, b^5) $0 = t_s < M < t_e < T$.

The boundary solutions of b^1 and b^2 are obviously the same as those of a^3 and a^1. When $M = t_e$ is inserted in *Eq. (42)* to *Eq. (44)*, b3 is solved. The point $(t_{e23}, t_{s23}, T_{24})$ is feasible only if it conforms to the constraint $0 < t_s < t_e < T$. To examine b^4, *Eq. (42)* and *Eq. (44)* must be solved with regard to the equations $M = t_e$ and $t_s = 0$. The point is acceptable only if $0 < t_e < T$. In the case of b^5, the equation $t_s = 0$ is placed in *Eq. (42), Eq. (44)* and *Eq. (45)*, and the point is feasible if $M < t_e < T$. The optimal solution in case 2 is achieved by comparing the profits of six possible feasible points.

5.3 | Optimal Solution under the Third Case ($0 < t_s \le t_e \le M < T$)

$Tsr_3 = Tr_3(t_e, t_s, T) + Ts(M, v)$,
s.t.
$0 \le t_s \le t_e \le M < T$.

Theorem 3.

1. For the given values of M, t_s and t_e, Tsr_3 is pseudo-concave to T.
2. For the given values of M, t_s and T, Tsr_3 is concave to t_e.
3. For the given values of M, t_e and T, Tsr_3 is concave to t_s.
4. For the given values of M, t_e and T, Tsr_3 is concave to M.

The proof for *Theorem 3* is the same as that for *Theorem 1*.

According to *Theorem 3*, Tsr_3 has an extremum point. As the equations $\frac{\partial Tsr_3}{\partial T} = 0$, $\frac{\partial Tsr_3}{\partial t_s} = 0$, $\frac{\partial Tsr_3}{\partial t_e} = 0$ and $\frac{\partial Tsr_3}{\partial M} = 0$ are simultaneously solved, the interior point $(M_3, t_{e3}, t_{s3}, T_3)$ obtained for the Tsr_3 is feasible if the constraint $0 \le t_s \le t_e \le M < T$ is true.

$$\begin{aligned}\frac{\partial Tsr_3}{\partial T} = {} & (6vR_i(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{-\theta(M-T)} + 6H(T(T\lambda + \alpha p_2 \\ & - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{\theta(T-t_e)} - 6HY\theta^2 e^{-\theta t_e} - 6((\alpha p_2 + \lambda t_e \\ & - B)\theta - \lambda)((MR_b v + c)\theta + h)e^{\theta t_e} - 6(p_1 - p_2)\theta\alpha((MR_b v + c)\theta + h)e^{\theta t_s} \\ & + (((M^3 R_e - 3MR_e t_s^2 + 2R_e t_s^3 - 3T^2 - 3t_s^2)p_2 - 2p_1 t_s^2(t_s R_e - 3/2Mr \\ & - 3/2))\lambda + ((3M^2 R_e - 6MR_e t_s + 3R_e t_s^2 - 6t_s)p_2^2 - 3p_1^2 t_s(-2MR_e \\ & + R_e t_s - 2))\alpha - 3B(M^2 R_e - 2MR_e t_s + R_e t_s^2 - 2t_s)p_2 + 6MYR_b v \\ & - 6BMp_1 R_e t_s + 3t_s p_1(R_e t_s - 2)B + 6Yc + 6A + 6O)\theta^3 + ((-3i(M^2 \\ & + T^2)v + (-3T^2 - 3t_e^2)H + 3t_e^2 h)\lambda + ((-6MvR_i - 6ht_s + 6t_e(h \\ & - H))p_2 + 6p_1(MR_b v + ht_s + c))\alpha + 6BM(R_i - R_b)v + (6Ht_e - 6ht_e \\ & - 6c)B + 6HY)\theta^2 + ((6(R_i - R_b)Mv + 6Ht_e - 6c)\lambda + ((6R_i v + 6H)p_2 \\ & + 6hp_1)\alpha - 6B(R_i v + H + h))\theta - 6\lambda(R_i v + H + h))/(6\theta^3 T^2).\end{aligned} \quad (46)$$

$$\begin{aligned}\frac{\partial Tsr_3}{\partial t_s} = {} & -(-((MR_b v + c)\theta + h)\alpha e^{\theta t_s} + ((-p_1 - p_2)\alpha - \lambda t_s + B)(-1 + (t_s - M)R_e)\theta + \\ & \alpha h)(p_1 - p_2)/(\theta T).\end{aligned} \quad (47)$$

$$\begin{aligned}\frac{\partial Tsr_3}{\partial t_e} = {} & (((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T-t_e)} - HY\theta^2 e^{-\theta t_e} - \theta(-\alpha p_2 - \lambda t_e + B)((MR_b v + c)\theta + \\ & h)e^{\theta t_e} + (h - H)(-\alpha p_2 - \lambda t_e + B)\theta - H\lambda)/(\theta^2 T).\end{aligned} \quad (48)$$

$$\begin{aligned}\frac{\partial Tsr_3}{\partial M} = {} & (2((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i(v\theta - g)e^{-\theta(M-T)} - 4(Mg + d/2)\theta R_b((-\alpha p_2 - \lambda t_e + \\ & B)\theta + \lambda)e^{\theta t_e} + 4(Mg + d/2)\theta^2 R_b(p_1 - p_2)\alpha e^{\theta t_s} + (-4MYR_b g - R_e p_2 \lambda M^2 + \\ & 2p_2 R_e(-\alpha p_2 + B)M - R_e t_s^2(p_1 - p_2)\lambda - 2dYR_b + 2R_e((-p_1 - p_2)\alpha + B)t_s(p_1 - \\ & p_2))\theta^3 + (((3\lambda M^2 + (4\alpha p_2 - 4B)M + 2(-\alpha p_2 - T\lambda/2 + B)T)R_i + 4MR_b(-\alpha p_1 + B))g + \\ & 2((M\lambda + \alpha p_2 - B)R_i + R_b(-\alpha p_1 + B))d)\theta^2 + (((-4M\lambda - 2\alpha p_2 + 2B)R_i + 4\lambda MR_b)g + \\ & 2\lambda d(R_b - R_i))\theta + 2\lambda g R_i)/(2\theta^3 T).\end{aligned} \quad (49)$$

The boundary solutions are then examined based on the following constraints:

c^1) $0 < t_s = t_e < M < T$. c^2) $0 < t_s < t_e < M = T$.

c^3) $0 = t_s = t_e < M = T$. c^4) $0 = t_s < t_e < M < T$.

In the case of c^1, $t_s = t_e$ is placed in *Eq. (46), Eq. (47)* and *Eq. (49)*, and the point $(M_{31}, t_{s31}, T_{31})$ is feasible only if $0 < t_s < M < T$. For c^2, the $M = T$ is inserted in *Eq. (46)* to *Eq. (48)*, the $(t_{e32}, t_{s32}, T_{32})$ is acceptable if $0 < t_s < t_e < T$. As for c^3, the relations $M = T$ and $t_s = t_e$ are inserted in *Eq. (46)* and *Eq. (47)*, and the point is acceptable only in conformity to the constraint $0 < t_s < T$. Moreover, the point obtained from the insertion of $t_s = 0$ in *Eq. (46), Eq. (48)* and *Eq. (49)* is feasible if $0 < t_e < M < T$. The final optimal solution is achieved by comparing the five feasible solutions.

5.4 | Optimal Solution under the Fourth Case ($0 \leq M \leq t_e \leq t_s < T$)

$\mathrm{Tsr}_4 = \mathrm{Tr}_4(t_e, t_s, T) + \mathrm{Ts}(M, v)$,
s.t.
$0 \leq M \leq t_e \leq t_s < T$.

Theorem 4.

1. For the given values of M, t_s and t_e, Tsr_4 is pseudo-concave to T.
2. For the given values of M, t_s and T, Tsr_4 is concave to t_e.
3. For the given values of M, t_e and T, Tsr_4 is concave to t_s.
4. For the given values of t_s, t_e and T, Tsr_4 is concave to M.

The proof of *Theorem 4* is the same as that of *Theorem 1*.

According to *Theorem 4*, Tsr_4 has an extremum point. As the equations $\frac{\partial \mathrm{Tsr}_4}{\partial T} = 0$, $\frac{\partial \mathrm{Tsr}_4}{\partial t_s} = 0$, $\frac{\partial \mathrm{Tsr}_2}{\partial t_e} = 0$ and $\frac{\partial \mathrm{Tsr}_2}{\partial M} = 0$ are simultaneously solved, the interior point $(M_4, t_{e4}, t_{s4}, T_4)$ obtained for the Tsr_4 is feasible if it satisfies the constraint $0 \leq M \leq t_e \leq t_s < T$.

$$\begin{aligned}\frac{\partial \mathrm{Tsr}_4}{\partial T} = {} & (-((R_i v - H)(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{\theta(T + 2t_e - t_s)})/2 - \\ & R_i((\alpha p_1 + \lambda t_e - B)\theta - \lambda)ve^{-\theta(M - 3t_e)} + e^{-\theta(M - 2t_e)} Y R_i \theta^2 v + ((T(T\lambda + \alpha p_2 - B)\theta^2 + \\ & (-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v + H)e^{\theta(T + t_s)})/2 - (R_i v + H)(((t_s - t_e)\lambda + \alpha(p_1 - p_2))\theta - \\ & \lambda)e^{\theta(t_e + t_s)} + R_i v(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{\theta(T + t_e)} + ((MYR_b v + \\ & (-p_2 T^2/2 + (t_s^2/2 + M^3 R_e/6)p_1 - p_2 t_s^2/2)\lambda + (p_1^2(M^2 R_e/2 + t_s) - p_2^2 t_s)\alpha + ((-t_s - \\ & M^2 R_e/2)p_1 + p_2 t_s)B + Yc + A + O)\theta^3 + ((-R_i(M^2 + T^2 + t_e^2 - t_s^2)\lambda/2 + ((p_1(t_s - \\ & M) - p_2 t_e)\alpha + B(t_e - t_s + M))R_i + MR_b(\alpha p_1 - B))v + ((-T^2/2 - t_e^2/2)H + \\ & t_e^2 h/2)\lambda + (((t_s - t_e)p_1 - p_2 t_s)H + p_1(ht_e + c))\alpha + (Bt_e + Y)H - B(ht_e + c))\theta^2 + \\ & ((((2t_e - t_s + M)R_i - MR_b)\lambda + 2R_i(\alpha p_1 - B))v + (Ht_e - c)\lambda + (H + h)(\alpha p_1 - B))\theta - \\ & \lambda(3R_i v + 2H + h))e^{2\theta t_e} - ((\alpha p_1 + \lambda t_e - B)\theta - \lambda)((MR_b v + c)\theta + h)e^{3\theta t_e} - \\ & \theta^2 e^{\theta t_e} Y(R_i v + H))e^{-2\theta t_e}/(\theta^3 T^2).\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial \mathrm{Tsr}_4}{\partial t_s} = {} & (-((-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v + H)e^{\theta(T - 2t_e + t_s)} - 2(R_i v + H)((t_e - t_s)\lambda - \\ & \alpha(p_1 - p_2))\theta e^{-\theta(t_e - t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)(-vR_i + H)e^{\theta(T - t_s)} + 2(p_1 - \\ & p_2)((-p_1 - p_2)\alpha - \lambda t_s + B)\theta^2 + (2v(-\alpha p_1 - \lambda t_s + B)R_i - 2\alpha H(p_1 - p_2))\theta + \\ & 2\lambda v R_i)/(2\theta^2 T).\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{\partial \mathrm{Tsr}_4}{\partial t_e} = {} & e^{-2\theta t_e}(-R_i \theta v(-\alpha p_1 - \lambda t_e + B)e^{-\theta(M - 3te)} - ((t_s - t_e)\lambda + \alpha(p_1 - p_2))\theta(R_i v + \\ & H)e^{\theta(te + t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v + H)e^{\theta(T + t_s)} + R_i v((-T\lambda - \alpha p_2 + B)\theta + \\ & \lambda)e^{\theta(T + t_e)} + ((-v(-\alpha p_2 - \lambda t_e + B)R_i + (h - H)(-\alpha p_1 - \lambda + B))\theta - 2\lambda(R_i v + \\ & H/2))e^{2\theta t_e} - (((M^2 bg + MR_b d + c)\theta + h)(-\alpha p_1 - \lambda t_e + B)e^{3\theta t_e} + Ye^{\theta t_e}\theta(R_i v + \\ & H))\theta)/(\theta^2 T).\end{aligned} \tag{52}$$

$$\begin{aligned}\frac{\partial \mathrm{Tsr}_4}{\partial M} = {} & e^{-2\theta t_e}(g((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i e^{\theta(T + 2t_e - t_s)}/2 + ((-\alpha p_1 - \lambda t_e + B)\theta + \\ & \lambda)R_i(v\theta - g)e^{-\theta(M - 3t_e)} + \theta^2 R_i(v\theta - g)Ye^{-\theta(M - 2t_e)} + gR_i(((t_s - t_e)\lambda + \alpha(p_1 - p_2))\theta - \\ & \lambda)e^{\theta(t_e + t_s)} - g((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i e^{\theta(T + t_e)} - g((-T\lambda - \alpha p_2 + B)\theta + \\ & \lambda)R_i e^{\theta(T + t_s)}/2 + ((-2MYR_b g - \lambda M^2 p_1 R_e/2 + p_1 R_e(-\alpha p_1 + B)M - dYR_b)\theta^3 + \\ & ((((t_e^2/2 - t_s^2/2 + (3M^2)/2 - T^2/2)\lambda + (2\alpha p_1 - 2B)M + (-p_1 t_s + p_2(t_e - T))\alpha - \\ & B(t_e - t_s - T))g - d(-M\lambda - \alpha p_1 + B))R_i + 2(Mg + d/2)R_b(-\alpha p_1 + B))\theta^2 +\end{aligned} \tag{53}$$

$((((-2t_e + t_s - M)\lambda - 2\alpha p_1 + 2B)g - \lambda d)R_i + 2(Mg + d/2)\lambda R_b)\theta + 3\lambda gR_i)e^{2\theta t_e} - 2\theta((Mg + d/2)((-\alpha p_1 - \lambda t_e + B)\theta + \lambda)R_b e^{3\theta t_e} - \theta e^{\theta t_e}YgR_i/2))/(\theta^3 T)$.

The boundary solutions are then examined based on the following constraints:

$d^1)M = 0, 0 < t_e < t_s < T$, $\quad d^2)0 < M = t_e < t_s < T$.

In the case of d^1, $M = 0$ is placed in *Eq. (50)* to *Eq. (52)*, and the point $(0, t_{e41}, t_{s41}, T_{41})$ is feasible only if the constraint $0 < t_e < t_s < T$ is met. For d^2, concerning $M = t_e$, the point obtained through solving *Eq. (50)* to *Eq. (52)* is feasible if $0 < t_e < t_s < T$ is satisfied. The optimal solution results from comparing the profits of three feasible points.

5.5 | Optimal Solution under the Fifth Case ($0 < t_e \le t_s \le M \le T$)

$Tsr_5 = Tr_5(t_e, t_s, T) + Ts(M, v)$,

s.t.

$0 < t_e \le t_s \le M \le T$.

Theorem 5.

1. For the given values of M, t_e and t_s, Tsr_5 is pseudo-concave to T.
2. For the given values of M, t_s and T, Tsr_5 is concave to t_e.
3. For the given values of M, t_e and T, Tsr_5 is concave to t_s.
4. For the given values of t_s, t_e and T, Tsr_5 is concave to M.

The proof for *Theorem 5* is the same as that for *Theorem 1*.

Based on *Theorem 5*, Tsr_5 has an extremum point. Once the equations $\frac{\partial Tsr_5}{\partial T} = 0$, $\frac{\partial Tsr_5}{\partial t_s} = 0$, $\frac{\partial Tsr_5}{\partial t_e} = 0$ and $\frac{\partial Tsr_5}{\partial M} = 0$ are solved simultaneously, the interior point $(M_5, t_{e5}, t_{s5}, T_5)$ obtained for the Tsr_5 is feasible if it satisfies the constraint $0 \le M \le t_e \le t_s < T$.

$$\frac{\partial Tsr_5}{\partial T} = e^{-2\theta t_e}(R_i v(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{(-2M+T+2t_e+t_s)\theta}/2 - R_i(((t_s - M)\lambda + \alpha(p_1 - p_2))\theta - \lambda)ve^{-\theta(M-2t_e-t_s)} + ((T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v + H)e^{\theta(T+2t_e-t_s)})/2 + H(T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)e^{\theta(T+t_s)}/2 - H(((t_s - t_e)\lambda + \alpha(p_1 - p_2))\theta - \lambda)e^{\theta(t_e+t_s)} + ((((t_s{}^2/2 + M^3R_e/6)p_1 - p_2(T^2 + t_s{}^2)/2)\lambda + MYR_b v + (p_1{}^2(M^2R_e/2 + t_s) - p_2{}^2 t_s)\alpha + ((-t_s - M^2R_e/2)p_1 + p_2 t_s)B + Yc + A + O)\theta^3 + ((-R_i(M^2 + T^2)v/2 + (-T^2/2 - t_e{}^2/2)H + t_e{}^2h/2)\lambda + (((p_1(t_s - M) - p_2 t_s)R_i + p_1 MR_b)\alpha + BM(R_i - R_b))v + (((t_s - t_e)p_1 - p_2 t_s)\alpha + Bt_e + Y)H + (\alpha p_1 - B)(ht_e + c))\theta^2 + (((p_1 - p_2)Mv + Hte - c)\lambda + (\alpha p_1 - B)(R_i v + H + h))\theta - \lambda(2R_i v + 2H + h))e^{2\theta t_e} - ((\alpha p_1 + \lambda t_e - B)\theta - \lambda)((MR_b v + c)\theta + h)e^{3\theta t_e} - He^{\theta t_e}Y\theta^2)/(\theta^3 T^2). \tag{54}$$

$$\frac{\partial Tsr_5}{\partial t_s} = (-R_i((-T\lambda - \alpha p_2 + B)\theta + \lambda)ve^{(-2M+T+t_s)\theta} - ((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T-2t_e+t_s)} + 2R_i((t_s - M)\lambda + \alpha(p_1 - p_2))v\theta e^{-\theta(M-t_s)} - 2((t_e - ts)\lambda - \alpha(p_1 - p_2))\theta He^{-\theta(t_e-t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)(R_i v + H)e^{\theta(T-t_s)} + 2(p_1 - p_2)(((-p_1 - p_2)\alpha - \lambda t_s + B)\theta - (R_i v + H)\alpha)\theta)/(2\theta^2 T). \tag{55}$$

$$\frac{\partial Tsr_5}{\partial t_e} = e^{-2\theta t_e}(-\theta H((t_s - t_e)\lambda + \alpha(p_1 - p_2))e^{\theta(t_e+t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T+t_s)} - \theta(-\alpha p_1 - \lambda t_e + B)((MR_b v + c)\theta + h)e^{3\theta t_e} + ((h - H)(-\alpha p_1 - \lambda t_e + B)\theta - H\lambda)e^{2\theta t_e} - He^{\theta t_e}Y\theta^2)/(\theta^2 T). \tag{56}$$

$$\frac{\partial Tsr_5}{\partial M} = (((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i(v\theta - g/2)e^{(-2M+T+2t_e+t_s)\theta} - (v((t_s - M)\lambda + \alpha(p_1 - p_2))\theta^2 - g((t_s - M)\lambda + \alpha(p_1 - p_2))\theta + \lambda g)R_i e^{-\theta(M-2t_e-t_s)} - g((-T\lambda - \alpha p_2 + \tag{57}$$

$B)\theta + \lambda)R_i e^{\theta(T+2t_e-t_s)}/2 + ((-2MYR_b g - \lambda M^2 p_1 R_e/2 + p_1 R_e(-\lambda p_1 + B)M - dYR_b)\theta^3 + (((((3M^2)/2 - T^2/2)\lambda + (2\alpha p_1 - 2B)M + (-p_1 t_s + p_2(t_s - T))\alpha + BT)g - d(-M\lambda - \alpha p_1 + B))R_i + 2(Mg + d/2)R_b(-\alpha p_1 + B))\theta^2 + (((-2M\lambda - \alpha p_1 + B)g - \lambda d)R_i + 2(Mg + d/2)\lambda R_b)\theta + 2\lambda g R_i)e^{2\theta t_e} - 2(Mg + d/2)((-\alpha p_1 - \lambda t_e + B)\theta + \lambda)\theta R_b e^{3\theta t_e})e^{-2\theta t_e}/(\theta^3 T).$

The boundary solutions are then examined based on the following constraints:

$e^1)\ 0 < t_e < t_s = M < T,\ \ e^2)\ 0 < t_e < t_s < M = T.$

In the case of e^1, the relation $t_s = M$ is placed in *Eq. (54)* and *Eq. (56)*, and the point $(t_{e51}, t_{s51}, T_{51})$ is feasible if it does not violate the constraint $0 < t_e < t_s < T$. For the case e^2, the $M = T$ is inserted in *Eq. (54)* to *Eq. (56)* to obtain the point $(t_{e52}, t_{s52}, T_{52})$. This point can be accepted if the constraint $0 < t_e < t_s < T$ is satisfied. The optimal solution is achieved by comparing the profits of three feasible points.

5.6 | Optimal Solution under the Sixth Case ($0 \le t_e \le M \le t_s \le T$)

$Tsr_6 = Tr_6(t_e, t_s, T) + Ts(M, v).$
s.t.
$0 < t_e \le M \le t_s \le T.$

Theorem 6.

1. For the given values of M, t_s and t_e, Tsr_6 is pseudo-concave to T.
2. For the given values of M, t_s and T, Tsr_6 is concave to t_e.
3. For the given values of M, t_e and T, Tsr_6 is concave to t_s.
4. For the given values of t_s, t_e and T, Tsr_6 is concave to M.

The proof for *Theorem 6* is the same as that for *Theorem 1*.

Based on *Theorem 6*, Tsr_6 has an extremum point. Once the equations $\frac{\partial Tsr_6}{\partial T} = 0, \frac{\partial Tsr_6}{\partial t_s} = 0, \frac{\partial Tsr_6}{\partial t_e} = 0$ and $\frac{\partial Tsr_6}{\partial M} = 0$ are solved simultaneously, the interior point $(M_6, t_{e6}, t_{s6}, T_6)$ obtained for the Tsr_6 is feasible if it is true in the constraint $0 < t_e \le M \le t_s < T$.

$$\frac{\partial Tsr_6}{\partial T} = ((T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i v e^{-\theta(M-T-2t_e)} + ((T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T+2t_e-t_s)})/2 + ((T(T\lambda + \alpha p_2 - B)\theta^2 + (-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T+t_s)})/2 - (((t_s - t_e)\lambda + \alpha(p_1 - p_2))\theta - \lambda)He^{\theta(t_e+t_s)} + ((((t_s{}^3 R_e/3 + (-MR_e/2 - 1/2)t_s{}^2 + M^3 R_e/6 - T^2/2)p_2 - ((t_s R_e - 3MR_e/2 - 3/2)t_s{}^2 p_1)/3)\lambda + ((t_s{}^2 R_e/2 + (-MR_e - 1)t_s + M^2 R_e/2)p_2{}^2 - p_1{}^2 t_s(-2MR_e + R_e t_s - 2)/2)\alpha + ((-t_s{}^2 R_e/2 + (MR_e + 1)t_s - M^2 R_e/2)p_2 + t_s p_1(-2MR_e + R_e t_s - 2)/2)B + MYR_b v + Yc + A + O)\theta^3 + ((-R_i(M^2 + T^2)v/2 + (-T^2/2 - t_e{}^2/2)H + t_e{}^2 h/2)\lambda + ((-p_2 t_s - p_1(t_e - t_s))\alpha + Bt_e + Y)H + (-R_i p_2 vM + p_1(Mbv + ht_e + c))\alpha - (-(R_i - R_b)Mv + t_e h + c)B)\theta^2 + (((R_i - R_b)Mv + Ht_e - c)\lambda + (\alpha p_1 - B)H + (R_i p_2 v + hp_1)\alpha - B(R_i v + h))\theta - \lambda(R_i v + 2H + h))e^{2\theta t_e} - ((MR_b v + c)\theta + h)((\alpha p_1 + \lambda t_e - B)\theta - \lambda)e^{3\theta t_e} - He^{\theta t_e} Y\theta^2)e^{-2\theta t_e}/(\theta^3 T^2). \tag{58}$$

$$\frac{\partial Tsr_6}{\partial t_s} = (-((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T-2t_e+t_s)} + 2\theta(-\alpha p_2 + (t_s - t_e)\lambda + \alpha p_1)He^{-\theta(t_e-t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T-t_s)} + 2\theta(p_1 - p_2)((-\alpha p_1 - \alpha p_2 - \lambda t_s + B)(1 + (M - ts)R_e)\theta - H\alpha))/(2\theta^2 T). \tag{59}$$

$$\frac{\partial Tsr_6}{\partial t_e} = e^{-2\theta te}(-\theta H((t_s - t_e)\lambda + \alpha(p_1 - p_2))e^{\theta(t_e + t_s)} + ((-T\lambda - \alpha p_2 + B)\theta + \lambda)He^{\theta(T + t_s)} - \theta(-\alpha p_1 - \lambda t_e + B)((MR_b v + c)\theta + h)e^{3\theta t_e} + ((h - H)(-\alpha p_1 - \lambda t_e + B)\theta - H\lambda)e^{2\theta t_e} - He^{\theta t_e}Y\theta^2)/(\theta^2 T). \quad (60)$$

$$\frac{\partial Tsr_6}{\partial M} = (((-T\lambda - \alpha p_2 + B)\theta + \lambda)R_i(v\theta - g)e^{-\theta(M - T - 2t_e)} + ((-2MYR_b g - R_e(p_2 M^2 + t_s{}^2(p_1 - p_2))\lambda/2 + p_2 R_e(-\alpha p_2 + B)M - dYR_b + R_e((-p_1 - p2)\alpha + B)t_s(p_1 - p_2))\theta^3 + (((((3M^2)/2 - T^2/2)\lambda - 2(-\alpha p_2 + B)(M - T/2))R_i + 2MR_b(-\alpha p_1 + B))g + ((M\lambda + \alpha p_2 - B)R_i + R_b(-\alpha p_1 + B))d)\theta^2 + (((-2M\lambda - \alpha p_2 + B)R_i + 2\lambda MR_b)g + \lambda d(R_b - R_i))\theta + \lambda gR_i)e^{2\theta t_e} - 2(Mg + d/2)((-\alpha p_1 - \lambda t_e + B)\theta + \lambda)\theta R_b e^{3\theta t_e})e^{-2\theta t_e}/(\theta^3 T). \quad (61)$$

The investigation of the boundary solutions in case 6 is avoided here because they are the same as those in the previous cases. In this case, the optimal solution originates from the only interior feasible point.

6 | Numerical Solution

This section evaluates the proposed model concerning the input parameters in *Table 3*. The parameters are adopted based on Kaya and Polat's [6] research and the logical values for the new parameters used in the model. To evaluate the optimality conditions of each case, six different examples generated by changing the values of some parameters are introduced. In each example, the values of the decision variables and the profits of the supply chain members are calculated. The case with the highest profit is selected as an optimal one.

Table 3. Input parameters.

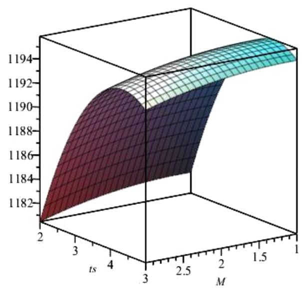
c	O	A	H	h	θ	λ	α	B
1	2000	5000	0.5	0.1	0.09	4	0.8	100
Y	l	g	p_2	p_1	R_b	R_e	R_i	
100	0.6	10	48	60	0.1	0.01	0.15	

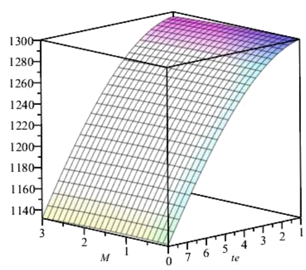
Example 1.

Table 4. Optimal values in Example 1.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case 1*	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
Case 2	382.225	809.267	1191.492	3.798	6.254	3.798	7.844	458.956	358.956
Case 3	423.661	758.965	1182.626	6.082	6.082	4.451	7.664	438.790	338.790
Case 4	253.131	935.236	1188.367	1.096	6.008	6.008	7.586	410.054	310.054
Case 5	375.060	801.601	1176.660	5.888	5.888	5.888	7.462	404.300	304.300
Casem6	375.060	801.601	1176.660	5.888	5.888	5.888	7.462	404.300	304.300

Case 1 with the value of $(M = 1.091, t_e = 6.248, t_s = 4.360, T = 7.838)$, and the profit of 1195.877 is an optimal case (*Table 4*). According to this case and the optimal point, the supply chain manager should discount the price at time 4.360 after the trade credit expiration at time 1.091 and before ending the RW inventory at time 6.248. The profits of the food producer and the supermarket are 294.545 and 901.331, respectively. *Fig. 2* shows the concavity of the supply chain function at the optimal point.





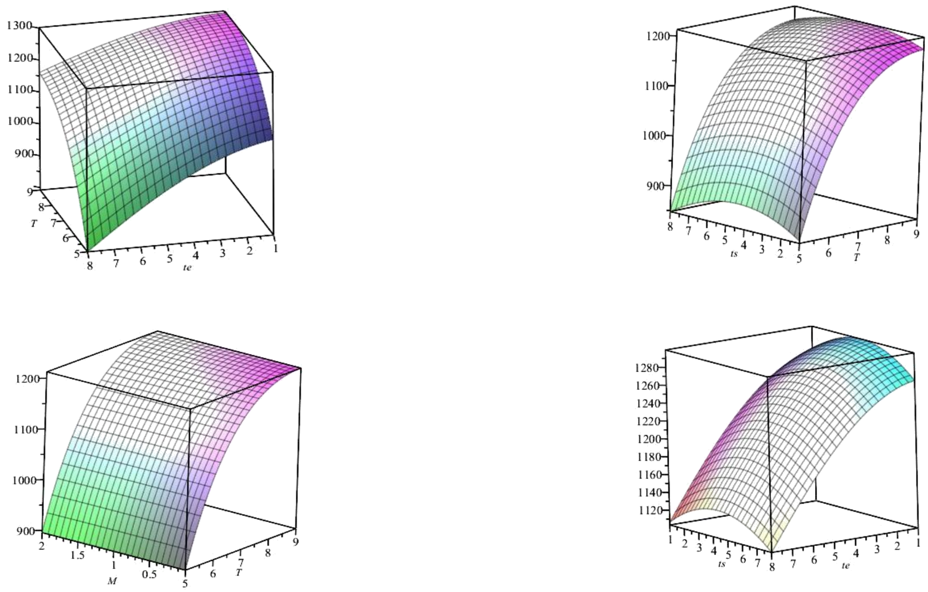


Fig. 2. Concavity of the supply chain profit function at point $(\mathbf{M = 1.091, te = 6.248, ts = 4.360, T = 7.838})$.

Example 2. In this example, with regard to the parameters in *Example 1*, except for the change of $\mathrm{R_e}$ from 0.01 to 0.15, the optimality shifts from case 1 to case 3 with a profit of 1287.257(*Table 5*). Based on case 3 and the optimal solution, the manager should adopt a trade credit period ($\mathrm{M = 8.033}$) which is longer than the RW inventory depletion time ($\mathrm{t_e = 6.434}$) and the discount time ($\mathrm{t_s = 4.504}$). As the optimal solution suggests, the trade credit period is considered as long as the cycle, and the price is discounted during the RW inventory consumption.

Table 5. Optimal values in Example 2.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case 1	391.694	858.880	1250.574	6.242	6.242	6.242	7.831	421.216	321.216
Case 2	451.046	808.347	1259.393	6.495	6.495	4.442	8.097	465.394	365.394
Case 3*	486.618	800.639	1287.257	8.033	6.434	4.504	8.033	460.651	360.651
Case 4	391.694	858.880	1250.574	6.242	6.242	6.242	7.831	421.216	321.216
Case 5	391.694	858.880	1250.574	6.242	6.242	6.242	7.831	421.216	321.216
Case 6	427.725	851.566	1279.291	7.781	6.194	6.194	7.781	418.963	318.963

Example 3. Considering the parameters in *Example 1* and the change of $\mathrm{R_i}$ from 0.15 to 0.25, the optimality shifts from case 1 to case 3 with a profit of 937.779 and the optimal value of ($\mathrm{M = 3.256, t_e = 6.254, t_s = 1.169, T = 8.126}$) (*Table 6*).

Table 6. Optimal values in Example 3.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case 1	289.989	644.027	934.016	1.645	6.252	1.645	8.123	442.822	342.822
Case 2*	340.903	595.876	936.779	3.256	6.254	1.169	8.126	448.765	348.765
Case 3	410.862	518.932	929.794	6.143	6.143	1.280	8.004	441.242	341.242
Case 4	230.579	648.880	879.459	3.230	5.784	5.784	7.612	353.706	253.706
Case 5	283.873	591.583	875.456	5.718	5.718	5.718	7.540	351.164	251.164
Case 6	283.873	591.583	875.456	5.718	5.718	5.718	7.540	351.164	251.164

Example 4. If all the parameters in *Example 1* remain constant and λ changes from 4 to 5, as well as R_i from 0.15 to 0.1, cases 1 and 4 will be optimal with a profit of 1634.217 and the value of ($\mathrm{M = 0, t_e = 5.493, t_s = 5.493, T = 6.918}$) (*Table 6*). In this value, $\mathrm{M = 0}$ means that no trade credit is provided, and the purchase cost should be settled on the spot. Since $\mathrm{t_s = t_e = 5.493}$, the RW inventory is sold at the prime price, and the OW inventory is sold at the marked-down price of 48.

Table 7. Optimal values in Example 4.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case 1*	295.182	1339.035	1634.217	0.000	5.493	5.493	6.918	449.133	349.133
Case 2	462.816	1153.016	1615.832	5.381	5.381	5.381	6.801	442.414	342.414
Case 3	462.816	1153.016	1615.832	5.381	5.381	5.381	6.801	442.414	342.414
Case 4*	295.182	1339.035	1634.217	0.000	5.493	5.493	6.918	449.133	349.133
Case 5	462.816	1153.016	1615.832	5.381	5.381	5.381	6.801	442.414	342.414
Case 6	462.816	1153.016	1615.832	5.381	5.381	5.381	6.801	442.414	342.414

Example 5. Considering *Example 4*, if R_e changes from 0.01 to 0.09, l from 0.6 to 0.1, and Y from 100 to 200, case 6 will be optimal. Following case 6 and the optimal solution, the purchase cost is paid at the end of the sale cycle $(M = T = 7.173)$. Also, the price is discounted at the time $t_s = 5.193$ after the RW inventory depletion time 4.783 (*Table 8*).

Table 8. Optimal values in Example 5.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case 1	331.433	1317.245	1648.678	4.532	4.532	4.532	7.369	490.419	290.419
Case 2	331.433	1317.245	1648.678	4.532	4.532	4.532	7.369	490.419	290.419
Case 3	332.554	1350.514	1683.068	7.363	4.526	4.526	7.363	490`.070	290.070
Case 4	389.730	1268.668	1658.398	5.042	5.042	5.889	7.004	521.874	321.874
Case 5	427.378	1245.419	1672.797	6.789	5.343	6.789	6.789	540.124	340.124
Case 6*	361.429	1323.912	1685.341	7.173	4.783	5.193	7.173	505.959	305.959

Example 6. With *Example 5* in mind, cases 5 and 6 will be optimal simultaneously if λ and Y are equal to 3 and 200, respectively. Since $t_s = T$ is at the optimal point, the supermarket has to adopt the fixed pricing strategy and sell all the products at $p_1 = 60$ (*Table 9*).

Table 9. Optimal values in Example 6.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
Case1	443.916	1954.002	2397.918	6.785	6.785	6.785	9.367	678.630	478.630
Case2	443.916	1954.002	2397.918	6.785	6.785	6.785	9.367	678.630	478.630
Case3	444.046	1994.588	2438.634	9.343	6.761	6.761	9.343	676.932	476.932
Case4	519.499	1930.123	2449.622	7.237	7.237	8.556	8.556	710.032	510.032
Case5*	519.660	1952.173	2471.833	8.538	7.219	8.538	8.538	708.771	508.771
Case6*	519.660	1952.173	2471.833	8.538	7.219	8.538	8.538	708.771	508.771

7 | Sensitivity Analysis

In this section, sensitivity analyses are conducted to investigate the model behavior as affected by the variation of the parameters. To this end, the variations of the demand-affecting parameters are first examined to check how they impact the decision variables and the profit of the supply chain members (*Table 10*).

Table 10. Sensitivity analysis of the decision variables impacted by the demand-affecting parameters.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
B=95	213.700	506.608	720.307	1.150	6.006	1.288	8.304	393.682	293.682
100	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
105	378.788	1343.605	1722.393	1.117	6.218	6.218	7.440	503.425	403.425
α =0.7	326.571	1171.902	1498.474	1.108	6.157	6.157	7.542	466.535	366.535
0.8	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.9	285.623	658.667	944.290	1.157	6.297	1.105	8.173	451.935	351.935
λ =3	345.967	1041.436	1387.402	1.033	7.198	6.475	8.495	518.336	418.336
4	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
5	236.272	788.017	1024.289	1.129	5.368	3.318	7.261	387.972	287.972
p_1=57	299.009	898.096	1197.105	1.072	6.243	5.130	7.832	454.729	354.729
60	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
63	296.747	892.702	1189.449	1.120	6.279	3.604	7.870	452.898	352.898
p_2=45	295.109	893.772	1188.882	1.078	6.365	5.130	7.850	451.939	351.939
48	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
51	284.747	914.573	1199.320	1.107	6.070	3.604	7.775	440.506	340.506

The large initial market (B) mainly involves more demand and sales. Therefore, to meet the demand, the supermarket raises the order quantity (Q_0). The more the sales, the later the markdown is. It means that the markdown time (t_s) shifts to the end of the sale cycle. The variation of B induces no specific trend of change in M and t_e, but a rise in B causes a decrease of T. Obviously, the food producer and the supermarket consider larger initial markets more desirable.

An increase in one of the α, λ, p_1 and p_2 parameters lead to a reduction in demand and sales. Once the demand decreases, the price is marked down sooner to lessen the adverse effect of the demand reduction. As a result, t_s moves closer to the beginning of the sale cycle. Moreover, T and t_e increase in response to the rise of α and p_1 but decrease in response to the rise of λ and p_2. As for M, its variation correlates positively to the changes of p_1, p_2 and λ. In response to the variation of α, however, M shows no specific change trend. *Table 11* presents the effects of the cost parameters on the decision variables and the profits of the supply chain members.

Table 11. Sensitivity analysis of the decision variables versus the cost parameters.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
h=0.05	294.417	904.783	1199.199	1.0917	6.245	4.360	7.834	450.559	350.559
0.1	294.545	901.331	1195.877	1.0914	6.248	4.360	7.838	450.783	350.783
0.15	294.674	897.880	1192.555	1.0912	6.252	4.360	7.841	451.007	351.007
θ=0.07	257.361	950.839	1208.200	1.161	6.097	4.292	7.925	421.090	321.090
0.09	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.1	314.229	875.191	1189.420	1.061	6.310	4.397	7.791	466.148	366.148
c=0.5	327.028	897.691	1224.719	1.094	6.257	4.175	7.846	453.922	353.922
1	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
2	229.974	908.748	1138.722	1.088	6.229	4.752	7.817	443.859	343.859
A=4500	282.532	978.476	1261.009	1.111	5.941	4.360	7.516	430.982	330.982
5000	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
5500	304.912	828.414	1133.326	1.070	6.545	4.360	8.150	469.779	369.779
O=1800	315.893	905.711	1221.604	1.100	6.127	4.360	7.710	442.963	342.963
2000	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
2200	273.760	896.802	1170.562	1.083	6.368	4.360	7.964	458.474	358.474
g=7	109.369	1093.112	1202.482	0.576	6.277	4.314	7.868	453.292	353.292
10	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
13	468.954	720.445	1189.399	1.294	6.221	4.406	7.809	448.395	348.395
l=0.4	290.018	906.424	1196.442	1.529	6.250	4.354	7.840	450.983	350.983
0.6	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.8	288.272	907.225	1195.497	0.631	6.248	4.363	7.838	450.742	350.742
H=0.2	309.366	924.192	1233.558	1.080	6.493	3.995	8.095	471.568	371.568
0.5	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.7	283.024	889.086	1172.110	1.098	6.088	4.655	7.669	436.189	336.189

In terms of variation, M correlates negatively to h, θ, c, A, O and l and positively to g and H. The increase of h, θ, A, and O has induced a rise in the quantity of the order (Q_0), thus postponing the inventory depletion in the RW (t_e). Also, Q_0 and t_e are negatively correlated to c, l, g and H parameters. While h, A and O just slightly affect t_s, a rise in θ, c, g, l and H postpones the markdown time. An increase in the parameters, except for h, A and O, shortens the sale cycle T. A rise in the cost parameter negatively affects the total profit of the supply chain (Trs). *Table 12* shows how the variation of the rates of the received, charged interests, and the opportunity cost affect the decision variables and the profits of the supply chain members.

Table 12. Sensitivity analysis of the decision variables versus the rates of the paid and received interests.

Case	Ts	Tr	Trs	M	t_e	t_s	T	Q_0	S
R_i=0.12	263.790	936.253	1200.042	0.000	6.271	4.331	7.861	452.634	352.634
0.15	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.17	312.349	881.405	1193.753	1.733	6.235	4.373	7.824	449.750	349.750
R_c=0.008	292.539	903.302	1195.841	1.022	6.249	4.360	7.838	450.801	350.801
0.01	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.012	296.812	899.106	1195.918	1.170	6.248	4.360	7.837	450.761	350.761
R_b=0.08	333.565	864.117	1197.682	2.349	6.248	4.342	7.837	451.024	351.024
0.1	294.545	901.331	1195.877	1.091	6.248	4.360	7.838	450.783	350.783
0.12	263.308	932.023	1195.331	0.027	6.252	4.364	7.842	450.959	350.959

As can be seen, when R_i rises, M and t_s move closer to the end of the sale cycle, but T and t_e move toward the beginning of the cycle. An increased R_i is more profitable for the food producer, but it causes a reduction in the supermarket profit, order quantity and the total profit of the supply chain.

Also, the variation of t_e, t_s and T is not very sensitive to the change of R_e, but an increase of R_e causes a rise in M. Moreover, an increase of R_e diminishes the supermarket's profit but has a positive effect on the profit of the food producer and the total profit of the supply chain.

An increase in parameter R_b leads to a slight rise in t_e, t_s and T but to a minor decline in M and Trs.

7.1 | Managerial Insights

In the scenario of two warehouses, this study offers managers an efficient method for choosing the best course of action in an integrated inventory system for perishable goods. This study considers potential scenarios to investigate the impact of trade credit and markdown regulations on the overall profit of the supply chain simultaneously. The suggested paradigm is investigated using six numerical cases. The quantitative findings show that supply chain participants benefit from a trade credit. It is advised that the managers use a trade credit longer than the cycle length to boost revenues in response to an increase in the received interest rate. When interest rates are high, on the other hand, managers ought to fix the trade credit shorter than the cycle length. The management needs to pay close attention to how the cost of the investment opportunity affects earnings. The cost of goods should be paid in cash if the investment opportunity cost is large. As a trade credit is provided, the supermarket can increase the order quantity more than the OW capacity. This increases the profit, although the supermarket needs to rent another warehouse. A markdown policy helps managers appropriately deal with time-sensitive demands. Reducing the product price incentives discourages customers from buying perishable products before they lose their value. In contrast, a discount policy boosts demand and raises sales volume.

The managers should think about the timing of discounting. A price markdown should be offered to customers at an appropriate time during a sale cycle to maximize profit. When the demand is highly sensitive to time and price, it is suggested that the managers mark down the price before the inventory of the rented warehouse is completely depleted. This implies that the markdown policy should be applied sooner. Based on the obtained results, the integration of trade credits and markdown policies positively impacts the profit of supply chains. Therefore, it is suggested that managers use a trade credit as a financial source and a markdown policy as a demand incentive mechanism for inventories subject to deterioration. This model also helps to know how much trade credit to adopt and when to discount the price.

8 | Conclusion and Recommendations for Future Research

In this study, a two-warehouse integrated inventory model was designed under markdown and trade credit policies. Managing an inventory of perishable products is more challenging if their demand is time-dependent. This necessitates the correct choice of order and sales policies; otherwise, unwanted costs will be imposed by the waste of products. As a demand-stimulating policy, markdown pricing helps supermarkets enhance their sales.

In addition, as a common financing policy in supply chains, crediting allows supermarkets to pay for the purchased goods after a certain time. Saving the sales revenues in an account with a certain interest rate gives those markets a higher income. In this regard, the present study examined various cases based on the possible relationships among the sales cycle length, markdown pricing time of inventory depletion in RW, and credit deadline. Then, through integrated decision-making and with a set of theorems, optimal values were obtained for the decisions of the two members of the chain. The proposed framework was evaluated with numerical examples, and some managerial insights were provided through the sensitivity analysis of each parameter.

This research has a few limitations. Firstly, only the manufacturer's trade credit was considered, and the customers' trade credit was overlooked. Secondly, the deterioration rate was considered constant. Thirdly, no shortage was allowed. This study can, thus, be extended by a) considering the deterioration rate as a function of time, b) including a two-level trade credit, and c) taking demand backlogging into account.

Conflicts of Interest

All co-authors have seen and agree with the manuscript's contents, and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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Appendix A

Proof of (A.1): we define $Tsr_1 = \frac{D_1(T)}{D_2(T)}$:

$$D_1(T) = p_1Q_1 + p_2Q_2 - vQ_0 - A - HO + IE_1 - IP_1 \text{ and } D_2(T) = T \geq 0. \quad \textbf{(A.1)}$$

Taking the first-order and second-order partial derivatives of $D_1(T)$ w.r.t T, we have

$$D_1'(T) = -N(p_2, T)((-R_iv - H)e^{\theta(T-te)} + R_iv + \theta p_2 + H)/\theta. \quad \textbf{(A.2)}$$

$$D_1''(T) = ((N(p_2, T)\theta + \lambda)(R_iv + H)e^{\theta(T-te)} - (R_iv + \theta p_2 + H)\lambda)/\theta. \quad \textbf{(A.3)}$$

$Tsr_1 = \frac{D_1(T)}{D_2(T)}$ is a strictly pseudo-concave function in T when $D_1''(T) < 0$ [49].

For $D_1''(T) < 0$, the $\left((N(p_2, T)\theta + \lambda)(R_iv + H)e^{\theta(T-te)} - (R_iv + \theta p_2 + H)\lambda\right) < 0$ should be true.

Proof of (A.2): taking the second-order partial derivatives of Tsr_1 w.r.t t_s,

$$\frac{\partial^2 Tsr_1}{\partial\, t_s^2} = \frac{(p_1 - p_2)\left(\alpha R_ive^{-\theta(M-ts)} + \left((MR_bv + c)\theta + h\right)\alpha e^{\theta ts} - \lambda\right)}{T}, \quad \textbf{(A.4)}$$

when $\left(\alpha R_ive^{-\theta(M-ts)} + \left((MR_bv + c)\theta + h\right)\alpha e^{\theta ts} - \lambda\right) \leq 0$ is satisfied, we have $\frac{\partial^2 Tsr_1}{\partial\, t_s^2} \leq 0$.

Proof of (A.3): taking the second-order partial derivatives of Tsr_1 w.r.t t_e, we have

$$\frac{\partial^2 Tsr_1}{\partial\, t_e^2} = (vR_i(N(p_2, te)\theta + \lambda)e^{-\theta(M-te)} + (N(p_2, T)\theta - \lambda)(R_iv + H)e^{\theta(T-te)} + \theta^2Y(R_iv + H)e^{-\theta te} + (N(p_2, te)\theta + \lambda)((MR_bv + c)\theta + h)e^{\theta te} + \lambda(H - h))/(T\theta). \quad \textbf{(A.5)}$$

For $\frac{\partial^2 Tsr_1}{\partial\, t_e^2} \leq 0$, the$\Big(vR_i(N(p_2, te)\theta + \lambda)e^{-\theta(M-te)} + (N(p_2, T)\theta - \lambda)(R_iv + H)e^{\theta(T-te)} +$ $\theta^2Y(R_iv + H)e^{-\theta te} + (N(p_2, te)\theta + \lambda)\left((MR_bv + c)\theta + h\right)e^{\theta te} + \lambda(H - h)\Big) \leq 0$ should be met.

Proof of (A.4): taking the second-order partial derivatives of Tsr_1 w.r.t M, we have

$$\frac{\partial^2 Tsr_1}{\partial M^2} = \frac{U^1}{U^2}. \quad \textbf{(A.6)}$$

$$\begin{aligned} U^1 = ((N(p_2, te)\theta - \lambda)(v\theta - 2g)R_ie^{-\theta(M-te)} &+ (p_1 - p_2)(v\theta - 2g)\theta R_i\alpha e^{-\theta(M-ts)} \\ &- (v\theta - 2g)\theta^2 iYe^{-\theta M} + 2gR_b(N(p_2, te)\theta - \lambda)e^{\theta te} + 2\alpha\theta R_bg(p_1 - p_2)e^{\theta ts} \\ &+ (-2YR_bg - p_1R_eN(p_1, M)\theta)\theta^2 \\ &+ \left(\left((3M\lambda + 2\alpha p_1 - 2B)R_i + 2b(-\alpha p_1 + B)\right)g + dR_i\lambda\right)\theta \\ &+ 2g\lambda(R_b - R_i). \end{aligned} \quad \textbf{(A.7)}$$

$$U^2 = (T\theta^2). \quad \textbf{(A.8)}$$

If $U^1 \geq 0$, therefore Tsr_1 is concave to M.